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Slow light in moving media reaches a paradoxical regime when the flow speed of the medium approaches the group velocity of light. Pulses can penetrate a seemingly superluminal barrier and are reflected when the flow slows down.

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Imagine a wave packet propagating in a moving medium. Suppose first that the medium travels as a whole with uniform velocity. In this case we can construct a co-moving frame of coordinates where the medium is frozen. The observed speed of the pulse is the velocity of the wave packet in the co-moving frame transformed back to the laboratory frame. This mental exercise gives Einstein's addition theorem of velocities, which, in the case when both wave and medium propagate much slower than c , is reduced to the Galilean rule that the velocities of wave and medium simply add up.

Now imagine a non-uniformly moving medium, for example a gas flow through a nozzle. The addition theorem of velocities is still valid locally in each infinitesimal piece of the medium. Consequently, a wave packet moving against the current will never enter a region where the flow is faster than the wave velocity in the medium. So, if we send in a pulse of slow light [1–5] to move against the flow we would expect the pulse to freeze or to run backwards [6] when the speed of the medium exceeds the group velocity v_g . Yet exactly the opposite happens. The pulse will thrive beyond the anticipated superluminal barrier and will even become faster. On the other hand, the pulse will bounce back when, paradoxically, the flow slows down. This effect is sensitive to minute changes in flow speed and can be applied in precision measurements of flows with the present state of the art in slowing down light [1–5].

Let us examine the propagation of slow-light pulses in moving media. We assume that both the flow speed and the material parameters of the medium do not change significantly within the range of an optical wavelength. In this case we can describe light by the dispersion relation [7] between the wave vector \mathbf{k}' and the frequency ω' ,

$$k'^2 - \frac{\omega'^2}{c^2} - \chi(\omega') \frac{\omega'^2}{c^2} = 0, \quad (1)$$

in the locally co-moving frames of the medium indicated with primes. Slow light is supported by an extremely dispersive medium where the susceptibility χ reacts strongly and proportionally to the detuning between ω' and a resonance frequency ω_0 of the medium atoms,

$$\chi(\omega') = \frac{2c}{v_g} \frac{\omega' - \omega_0}{\omega_0}. \quad (2)$$

The group velocity v_g has been demonstrated [1–5] to be as low as a few meters per second. The specific mechanism to slow down light to such an impressive degree is not relevant for the analysis presented here. Slow light can be generated using Electromagnetically-Induced Transparency [8,9] or exploiting light-atom amplification in Bose-Einstein condensates [5]. Typically [1–5], the steep linear slope (2) of the susceptibility is only valid within a narrow frequency window

$$|\omega' - \omega_0| < \epsilon \frac{v_g}{c} \omega_0, \quad (3)$$

with ϵ being in the order of 10^{-3} . The moving medium creates a local Doppler shift of the frequency,

$$\omega' = \omega - \mathbf{u} \cdot \mathbf{k}, \quad \mathbf{k}' = \mathbf{k}, \quad (4)$$

in the realistic case of non-relativistic flow velocities \mathbf{u} .

Consider a pulse of slow light with a narrow range of frequencies ω near the carrier ω_0 . The Doppler detuning (4) due to the moving medium is only significant in the susceptibility. Consequently, we can approximate the dispersion relation (1) as

$$k^2 - k_0^2 - \frac{2k_0}{v_g} (\omega - \mathbf{u} \cdot \mathbf{k} - \omega_0) = 0. \quad (5)$$

with

$$k_0 = \frac{\omega_0}{c}. \quad (6)$$

The group velocity \mathbf{v} of the pulse is the derivative of the frequency ω with respect to the wave vector \mathbf{k} , in analogy to the velocity of a particle with Hamiltonian $\hbar\omega$ and momentum $\hbar\mathbf{k}$. We obtain immediately from the dispersion relation (5)

$$\mathbf{v} = \frac{\partial \omega}{\partial \mathbf{k}} = v_g \frac{\mathbf{k}}{k_0} + \mathbf{u}. \quad (7)$$

This formula is the addition theorem of velocities for slow light in moving media. Note that the group velocity v_g

at resonance frequency ω_0 and the flow speed \mathbf{u} do not simply add up. The effective group velocity in the co-moving frame is weighted and directed by the wave vector \mathbf{k} . This has profound consequences in non-uniformly moving media.

Consider, for simplicity, a one-dimensional model corresponding, for example, to a gas flow through a tube of varying width. According to the dispersion relation (5) the wave vector in 1D reduces to

$$k_{\pm} = k_0 \left(\pm \sqrt{1 + 2\frac{c}{v_g}\delta + \frac{u^2}{v_g^2}} - \frac{u}{v_g} \right) \quad (8)$$

with the detuning

$$\delta = \frac{\omega - \omega_0}{\omega_0}. \quad (9)$$

The \pm sign indicates propagation with (+) or against (−) the positive current u . Consequently, the group velocities are

$$v_{\pm} = \pm v_g \sqrt{1 + 2\frac{c}{v_g}\delta + \frac{u^2}{v_g^2}}. \quad (10)$$

In the case when the medium is moving uniformly with velocity u_0 we can construct a global co-moving frame where the pulse propagates with group velocity $\pm v_g$ at the carrier frequency ω_0 . When transforming back to the laboratory frame we must take into account the Doppler shift

$$\delta = \pm \frac{u_0}{c}, \quad (11)$$

and obtain from formula (10) the Galilean addition theorem of velocities,

$$v_0 = \pm v_g + u_0. \quad (12)$$

The situation is completely different when a slow light pulse propagates in a non-uniformly moving medium. Once launched, the pulse cannot change its frequency spectrum (unless the flow is not stationary). Consequently, the pulse cannot adapt to the flow by continuous Galilei transformations.

Imagine an asymptotically uniform flow with positive speed u_0 and a pulse propagating against the current with group velocity $-v_g + u_0$. When the flow speed changes from the initial u_0 to a varying u , the pulse will reach a turning point at a zero of the group velocity (10), *i.e.* at

$$u = v_g \sqrt{-2\frac{c}{v_g}\delta - 1} = v_g \sqrt{2\frac{u_0}{v_g} - 1}, \quad (13)$$

provided that u_0 exceeds $v_g/2$ or, equivalently, that the detuning δ reaches $-v_g/(2c)$. Note, however, that we

have not yet examined the validity range (3) of the dispersion relation (5). In terms of the effective group velocity (10) we obtain from Eqs. (4) and (7) the range

$$\left| \delta - \frac{u(v-u)}{v_g c} \right| < \epsilon \frac{v_g}{c}, \quad (14)$$

which gives at the turning point (13)

$$|u_0 - v_g| < \epsilon v_g. \quad (15)$$

Consequently, the initial value of the flow speed should lie in the vicinity of v_g . In this case the velocity (13) at the turning point is below u_0 . Pulses propagating against the current bounce back when the medium reduces speed. This effect establishes a qualitative change of the behavior of pulses, triggered by minute variations of the flow velocity, and hence even the narrow spectral width of slow light [1–5] gives a comfortable margin for the phenomenon to occur. Figure 1 shows a graphical method for analyzing slow light in moving media.

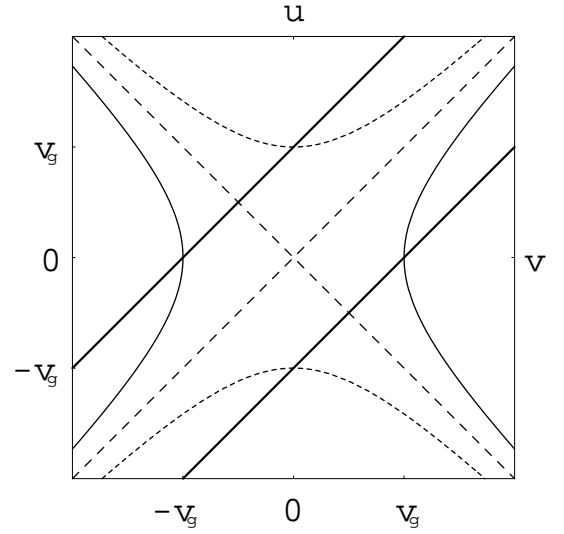


FIG. 1. Principal regimes of slow-light pulses in moving media. The figure shows the group velocity v versus the flow speed u for various values of the detuning δ . The dotted curves correspond to the common case of zero detuning when the pulse cannot freeze to $v = 0$. The solid curves represent the interesting case, $\delta = -v_g/c$, when the pulses reaches turning points outside the group-velocity gap between $u = \pm v_g$. The dashed lines show the marginal case of $\delta = -v_g/(2c)$. The plots are illustrations of Eq. (10) which has, however, a restricted validity range. In order to find the working points we employ the pair of thick diagonal lines shown in the figure as well. At these lines the Doppler detuning $\omega - uk - \omega_0$ vanishes for one-dimensional wave vectors k satisfying the dispersion relation (5), and the addition theorem (7) gives $v = \pm v_g + u$. Exactly at the working points where the diagonals cross the group-velocity curves the Galilean rule of velocity addition is obeyed, but not in a close vicinity of the points. This allows a pulse to bounce back when the flow slows down, as the figure clearly indicates, because the gap between the solid curves lies at the side of small medium velocities.

Our effect can serve to sense tiny variations in flow speed, as a form of slow-light sonar, because the phase of the reflected light depends on the turning point (13) for a given detuning. In order to scan a whole range of flow velocities, one should simply modify the group velocity in Electromagnetically-Induced Transparency by adjusting the intensity of the coupling beam [8].

So far, we have analyzed the geometrical optics of slow-light pulses, the dispersion relation and the group velocity, which determines the principal effects. Wave optics will tell us the finer details. In analogy to the correspondence between classical mechanics and wave mechanics, we regard the frequency ω and the wave vector \mathbf{k} as differential operators,

$$\omega = i\partial_t, \quad \mathbf{k} = -i\nabla, \quad (16)$$

and derive from the dispersion relation (5) the wave equation for the optical field φ ,

$$\left[\left(\mathbf{k} + k_0 \frac{\mathbf{u}}{v_g} \right)^2 - k_0^2 \left(1 + 2 \frac{c}{v_g} \frac{\omega - \omega_0}{\omega_0} + \frac{u^2}{v_g^2} \right) \right] \varphi = 0. \quad (17)$$

We see that the flow \mathbf{u} acts as an effective vector potential [7], similar to the behaviour of waves in non-dispersive media [10]. The flow also generates an attractive $-u^2/v_g^2$ potential [11] that happens to be proportional to the Bernoulli pressure of the fluid. When both medium and light move in only one dimension, say in the z direction, we can safely eliminate the vector potential by the gauge transformation

$$\varphi = \psi \exp \left(-ik_0 \int \frac{u}{v_g} dz \right). \quad (18)$$

We arrive at the Schrödinger equation

$$\left(\hbar\omega - \frac{\hbar^2 k^2}{2m} - U \right) \psi = 0 \quad (19)$$

with the effective mass m and the potential U ,

$$m = \frac{\hbar\omega_0}{v_g c}, \quad U = \frac{\hbar\omega_0}{2} \left(2 - \frac{v_g}{c} - \frac{u^2}{v_g c} \right). \quad (20)$$

In this way we can benefit from the accumulated experience with one-dimensional Schrödinger waves in scalar potentials. For instance, we can apply Heller's method [12] to analyze moving wave packets or use the efficient routines available for numerical simulations. Figure 2 illustrates the two typical regimes of slow-light wave packets, pulse acceleration and reflection.

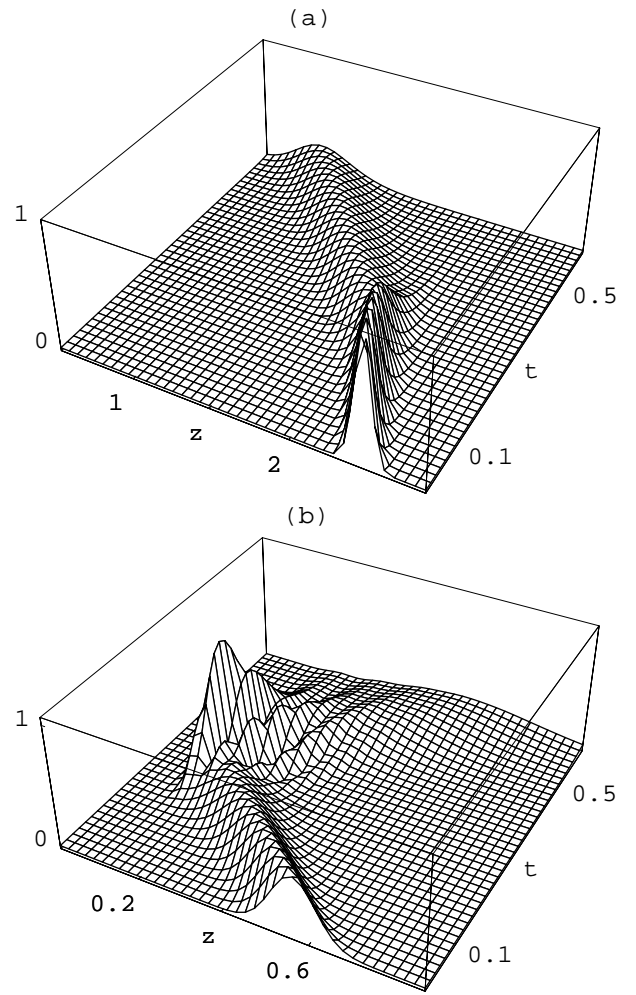


FIG. 2. Propagation of slow-light pulses in a moving medium. The medium supports a group velocity of $v_g = 300$ m/s at $\omega_0 = 3 \times 10^{15}$ Hz and moves with an initial flow speed $u_0 = 0.995v_g$. The pulses, with resonant carrier frequency $\omega = \omega_0(1 - u_0/v_g)$ and bandwidth 3 MHz, travel against the flow with a net group velocity of -1.5 m/s. (a) Pulse acceleration. At $z = 2$ mm, the flow speed increases by 1.15 cm/s and the pulse is accelerated to reach -3 m/s. (b) Pulse reflection when the medium velocity decreases. At the turning point the flow speed drops by a mere 3.8 mm/s, which illustrates the extreme sensitivity of slow-light pulses to variations of the medium velocity.

Figure 3 shows that even a uniform flow can reflect slow-light pulses when the group velocity v_g is spatially varying, a regime easily achieved in Electromagnetically-Induced Transparency [8] by modifying the intensity of the coupling beam. At the turning point the velocity (10) of the wave packet vanishes, which gives, for a uniform flow u_0 ,

$$v_g = -c\delta + \sqrt{c^2\delta^2 - u_0^2}. \quad (21)$$

For $\delta = -u_0/c$ the pulse is permanently frozen when the

motion of the medium exactly compensates the group velocity. A slightly different detuning, $\delta < -u_0/c$, will send the pulse back as soon as v_g reaches the critical value (21).

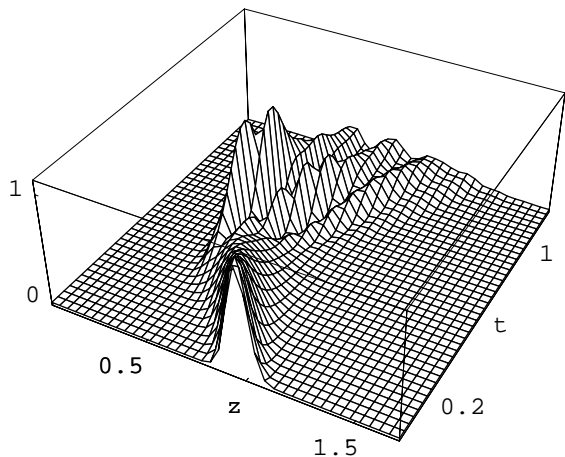


FIG. 3. Pulse reflection via variations of the group velocity v_g . The medium supports an initial v_g of 300 m/s and travels uniformly with $u_0 = 0.995 \times 300$ m/s. For a detuning $\delta = -u_0/c$ the pulse is frozen when v_g reaches u_0 . If we set δ to $-1.00001 \times u_0/c$ the wave packet bounces back when v_g drops by 0.16 m/s, as the figure illustrates.

Finally we remark that similar effects will also occur in a hot gas with a broad thermal spread of velocities, provided that the two beams involved in Electromagnetically-Induced Transparency, the coupling and the probe beam, are co-propagating [6]. Given a detuning δ_c of the coupling beam, only the atoms which are traveling with the right velocity, $u_0 = -c\delta_c$, reach the narrow EIT resonance. Consequently, slow light in hot gases is extremely sensitive to minute changes in the fraction of atoms moving at the critical speed u_0 .

To summarize, slow light in moving media shows a surprisingly paradoxical behavior when the flow approaches the group velocity of light. Pulses can not only penetrate a superluminal barrier but are also reflected when the flow slows down. Apart from illustrating puzzles with the addition theorem of velocities in moving media, our findings may lead to the slow-light analog of a flow sonar and they add an intriguing aspect to the fascinating field of slow-light optics [1–7,13].

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